Journal of Algebra **241,** 731–744 (2001)
doi:10.1006/jabr.2001.8767, available online at http://www.idealibrary.com on IDEAL®

Examples of QF Rings without Nakayama Automorphism and H-Rings without Self-Duality

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Communicated by Kent R. Fuller

Received July 25, 2000

J. Kado and K. Oshiro (*J. Algebra* **211** (1999), 384–408) proved the equivalence of (A) the existence of a Nakayama isomorphism for every basic left H-ring (Harada ring) and (B) the existence of a Nakayama automorphism for every basic QF ring. In this paper, by using examples of J. Kraemer ("Characterizations of the Existence of (Quasi-) Self-Duality for Complete Tensor Rings," Algebra Berichte 56, Verlag Reinhard Fischer, Munich, 1987), we give examples of QF rings without Nakayama automorphism and H-rings without self-duality. © 2001 Academic Press

Key Words: QF rings; H-rings; self-duality; Nakayama automorphism.

1. INTRODUCTION

By investigating the structure of H-rings deeply, Kado and Oshiro [5] proved that the following are equivalent:

(A) the existence of a Nakayama isomorphism for every basic left H-ring,

(B) the existence of a Nakayama automorphism for every basic QF ring,

(C) the existence of self-duality for every basic left H-ring.

On the other hand, Kraemer [6] constructed an example of a QF ring that does not have a weakly symmetric self-duality. In Section 2 we shall point out that Kraemer's example is an example of a QF ring without Nakayama automorphism (i.e., a counterexample of (B)) and give some other examples of QF rings without Nakayama automorphism. In Section 3, by giving

0021-8693/01 $35.00
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a necessary and sufficient condition for a special type of H-rings to have a self-duality (Theorem 3.1 and Corollary 3.1), we shall give concrete examples of H-rings without self-duality (i.e., a counterexample of (C)).

Throughout this paper, all rings have identity, all modules are unitary, and all homomorphisms are operated on the opposite side of scalars. For a module M, we denote, by $E(M)$, $J(M)$, $S(M)$, and $T(M)$, the injective hull, the radical, the socle, and the top of M (i.e., the factor module by its radical), respectively. For a ring R, we denote by R-Mod the category of left R-modules. For a left (resp. right) R-module X and a subset A of R, we denote by $r_X(A)$ (resp. $l_X(A)$) the annihilator of A in X.

2. QF RINGS WITHOUT NAKAYAMA AUTOMORPHISM

In this section, by presenting the example of Kraemer [6], we give examples of QF rings without Nakayama automorphism.

We begin by recalling terminologies of Morita duality. We say that a bimodule ${}_RU_S$ defines a *Morita duality* if ${}_RU_S$ is faithfully balanced and is an injective cogenerator on both sides. In particular, if $R = S$, we say that ${}_RU_R$ defines a *self-duality*. A ring R is said to have a *left Morita duality* if there exists a bimodule ${}_RU_S$ that defines a Morita duality. In this case, we say that R is *left Morita dual* to S. (See [9].)

Let ${}_RU_R$ define a self-duality. We say that ${}_RU_R$ defines a *weakly symmetric self-duality* if $T({}_RRe) \cong {}_R\mathrm{Hom}_R(T(eR_R), U_R)$ for every primitive idempotent e of R. Note that the condition of weakly symmetric self-duality is left−right symmetric. Since $\mathrm{Hom}_R(T(eR_R), U_R) \cong S({}_RUe)$, ${}_RU_R$ defines a weakly symmetric self-duality if and only if $T({}_RRe) \cong S({}_RUe)$ for every primitive idempotent e of R. A ring R is said to have a weakly symmetric self-duality if there exists a bimodule that defines a weakly symmetric self-duality. (See [6, p. 12].)

Remark 2.1. (1) As is well known, every artin algebra R has a weakly symmetric self-duality. Let K be the center of R and let $E = E(T(K))$ be the minimal injective cogenerator for K-Mod. Then $U = \mathrm{Hom}_K(R, E)$, which becomes an (R, R)-bimodule naturally, defines a weakly symmetric self-duality.

(2) A QF ring R is said to be weakly symmetric if the regular bimodule ${}_RR_R$ defines a weakly symmetric self-duality.

According to [5], we give the following definition. Let R be a basic semiperfect ring with left Morita duality. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of orthogonal primitive idempotents for R and let $S = \mathrm{End}_R(\bigoplus_{i=1}^n E(T(Re_i)))$ be the endomorphism ring of a minimal injective cogenerator for R-mod. Let f_i be the idempotent of S corresponding to the projection $\bigoplus_{i=1}^{n} E(T(Re_i)) \to E(T(Re_i))$. Then we call a ring isomorphism $\tau\colon R \to S$ a *Nakayama isomorphism* if $\tau(e_i) = f_i$ for each $i = 1, 2, \ldots, n$. By [7, p. 42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents. (See [5, Remark on p. 387].)

The following proposition shows that the existence of a weakly symmetric self-duality coincides with that of a Nakayama isomorphism.

PROPOSITION 2.1 ([4, Proposition 3.1]). *Let R be a basic semiperfect ring with left Morita duality and let ${}_RU$ be a minimal injective cogenerator for R-*Mod. *Then the following conditions are equivalent*:

(1) *R has a weakly symmetric self-duality.*

(2) *R has a Nakayama isomorphism.*

(3) *There exists a ring isomorphism $\tau\colon R \to \mathrm{End}({}_RU)$ such that $U\tau(e) \cong E(T(Re))$ for every primitive idempotent e of R.*

Proof. (1) ⇔ (3) is [4, Proposition 3.1] and (2) ⇔ (3) is clear. ∎

We recall the concepts of Nakayama permutations and Nakayama automorphisms for QF rings. Let ${}_RU_S$ be a bimodule that defines a Morita duality. Then by [1, Lemma 30.2] we have $S({}_RU) = S(U_S)$. So we simply denote this module by $S(U)$. Let $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ be basic sets of orthogonal primitive idempotents for R and S, respectively. Then there exists a permutation π on $\{1, 2, \ldots, n\}$ such that $S(e_iU) \cong T(f_{\pi(i)}S)$ and $S(Uf_{\pi(i)}) \cong T(Re_i)$ for each $i = 1, 2, \ldots, n$. In particular, when R is a QF ring with a basic set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents, there exists a permutation σ on $\{e_1, e_2, \ldots, e_n\}$ such that $S(e_iR) \cong T(\sigma(e_i)R)$ and $S(R\sigma(e_i)) \cong T(Re_i)$ for each $i = 1, 2, \ldots, n$. We call this permutation a *Nakayama permutation* of R. A ring automorphism τ of R is said to be a *Nakayama automorphism* of R if the restriction τ on $\{e_1, e_2, \ldots, e_n\}$ gives a Nakayama permutation of R. By Proposition 2.1, a basic QF ring R has a Nakayama automorphism if and only if R has a weakly symmetric self-duality.

We now give a condition to have a Nakayama automorphism for QF rings of a special type. For the sake of convenience, we denote by $[i]$ the least positive residue of an integer i modulo a positive integer m. In the proof of the next proposition (and that of Theorem 3.1), we shall use the useful concept of i-pair. Let e, f be primitive idempotents of R. A pair (eR, Rf) is called an *i-pair* if $S(eR) \cong T(fR)$ and $S(Rf) \cong T(Re)$ (see [2]). As is well known (e.g., see [1, Theorem 31.3]), when R is a one-sided artinian ring, for a primitive idempotent e of R, eR_R is injective if and only if there exists a primitive idempotent f of R such that (eR, Rf) is an i-pair. We shall use these results freely. Furthermore, we use the following terminology. Let $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ be ring homomorphisms and let ${}_AM_B$ and ${}_{A'}M'_{B'}$ be bimodules. An additive homomorphism $\phi\colon M \to M'$ is said to be (α, β)-*semilinear* if $\phi(amb) = \alpha(a)\phi(m)\beta(b)$ for all $a \in A$, $b \in B$, and $m \in M$.

PROPOSITION 2.2. *Let $A_1, A_2, \dots, A_m$ $(m \geq 2)$ be basic artinian rings and let ${}_{A_1}{U_1}_{A_2}, {}_{A_2}{U_2}_{A_3}, \dots, {}_{A_m}{U_m}_{A_1}$ be bimodules, each of which defines a Morita duality. Let*

$$R = \begin{pmatrix} A_1 & U_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & U_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1} & U_{m-1} \\ U_m & 0 & 0 & \cdots & 0 & A_m \end{pmatrix}$$

and define a ring structure on R by using ordinary matrix operations and by $U_iU_j = 0$ for $1 \leq i, j \leq m$. Then

(1) *R is a QF ring.*

(2) *R has a Nakayama automorphism if and only if for each $i = 1, 2, \dots, m$, there exists a ring isomorphism $\tau_i\colon A_i \to A_{[i+1]}$ such that $T(A_ie) \cong S(U_i\tau_i(e))$ for every primitive idempotent e of A_i, and there exists a $(\tau_i, \tau_{[i+1]})$-semilinear isomorphism $\phi_i\colon U_i \to U_{[i+1]}$.*

Proof. (1) By assumption, all of A_i have complete sets consisting of the same number (n, say) of orthogonal primitive idempotents. Thus, for each $i = 1, 2, \dots, m$, there exists a complete set $\{e_{ij} | 1 \leq j \leq n\}$ of orthogonal primitive idempotents for A_i. Then, since ${}_{A_i}{U_i}_{A_{[i+1]}}$ defines a Morita duality, there exists a permutation π_i on $\{1, 2, \dots, n\}$ such that $S(e_{ij}U_i) \cong T(e_{[i+1], \pi_i(j)}A_{[i+1]})$ and $S(U_ie_{[i+1], \pi_i(j)}) \cong T(A_ie_{ij})$ for $1 \leq j \leq n$. For each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, let f_{ij} be the idempotent of R such that its (i, i)-entry is e_{ij} and all other entries are zero, and let $f_i = \sum_{j=1}^n f_{ij}$ be the idempotent of R such that its (i, i)-entry is 1_{A_i} and all other entries are zero. Then $\{f_{ij} | 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a complete set of orthogonal primitive idempotents for R. By the definition of R, it is routine to check that $S(f_{ij}R) \cong T(f_{[i+1], \pi_i(j)}R)$ and $S(Rf_{[i+1], \pi_i(j)}) \cong T(Rf_{ij})$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus $(f_{ij}R, Rf_{[i+1], \pi_i(j)})$ become i-pairs and hence R is a QF ring.

(2) $(\Rightarrow)$ For the setting above, the Nakayama permutation of R is given by $\sigma\colon f_{ij} \mapsto f_{[i+1], \pi_i(j)}$. Assume that R has a Nakayama automorphism τ. Then $\tau(f_{ij}) = f_{[i+1], \pi_i(j)}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Since

$$\tau(f_i) = \tau\left(\sum_{j=1}^n f_{ij}\right) = \sum_{j=1}^n \tau(f_{ij}) = \sum_{j=1}^n f_{[i+1], \pi_i(j)} = f_{[i+1]},$$

it follows from $A_i \cong f_i R f_i$ and $U_i \cong f_i R f_{[i+1]}$ that τ induces ring isomorphisms τ_i: $A_i \to A_{[i+1]}$ and additive isomorphisms ϕ_i: $U_i \to U_{[i+1]}$. It is clear that $\tau_i(e_{ij}) = e_{[i+1], \pi_i(j)}$, so $T(A_i e_{ij}) \cong S(U_i e_{[i+1], \pi_i(j)}) = S(U_i \tau_i(e_{ij}))$. Thus $T(A_i e) \cong S(U_i \tau_i(e))$ for every primitive idempotent e of A_i. By the definitions of τ_i and ϕ_i, we see that ϕ_i is $(\tau_i, \tau_{[i+1]})$-semilinear.

($\Leftarrow$) Assume that τ_i and ϕ_i are given. We define τ: $R \to R$ via

$$\tau \begin{pmatrix} a_1 & u_1 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{m-1} \\ u_m & 0 & \cdots & a_m \end{pmatrix} = \begin{pmatrix} \tau_m(a_m) & \phi_m(u_m) & \cdots & 0 \\ 0 & \tau_1(a_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \phi_{m-2}(u_{m-2}) \\ \phi_{m-1}(u_{m-1}) & 0 & \cdots & \tau_{m-1}(a_{m-1}) \end{pmatrix}.$$

Then it is routine to check that τ is a ring automorphism of R such that $T(Rf) \cong S(R\tau(f))$ for every primitive idempotent f of R. Therefore by Proposition 2.1, R has a Nakayama automorphism. ∎

Remark 2.2. In the setting of Proposition 2.2, let $A = A_1 \times A_2 \times \cdots \times A_m$ be the product of rings and let $U = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ be the direct sum of $\mathbb{Z}$-modules. Then U becomes an (A, A)-bimodule naturally and ${}_AU_A$ defines a self-duality. Thus by [9, Theorem 10.7] the trivial extension of A by U is a QF ring. Indeed, R is isomorphic to the trivial extension.

We now present Kraemer's example. To state this, we need the following notation. Let C and D be rings. If ${}_CM_D$ is a bimodule, then let $M_1 = {}_CM_D$ and for $i = 2, 3, \ldots,$ define inductively

$$M_i = \begin{cases} {}_C\mathrm{Hom}_C\left({}_DM_{i-1_C}, {}_CC_C\right)_D & \text{if } i \text{ is odd}, \\ {}_D\mathrm{Hom}_D\left({}_CM_{i-1_D}, {}_DD_D\right)_C & \text{if } i \text{ is even}. \end{cases}$$

Example 2.1. By the results of [8] and [3], there exists an extension $C > D$ of division rings satisfying the following conditions (see [6, Theorems 6.1 and 6.2]).

(1) $\dim({}_DC) = 2$ and $\dim(C_D) = 3$.

(2) There exist ring isomorphisms λ: $D \to C$ and μ: $C \to D$.

(3) There exists a (λ, μ)-semilinear isomorphism ϕ: ${}_DC_{6C} \to {}_CC_{1D}$.

(4) $(a_1, a_2, a_3, a_4, a_5) = (3, 1, 2, 2, 1)$ and $(b_1, b_2, b_3, b_4, b_5) = (1, 3, 1, 2, 2)$,

where a_i and b_i denote the right and left dimensions of C_i, respectively. Then by [6, Lemma 6.3],

(5) The map $\psi\colon {}_C C_{7D} \to {}_D C_{2C}$ defined by $\psi(c_7)(c_1) = \mu(c_7(\phi^{-1}(c_1)))$ $(c_7 \in C_7,\ c_1 \in C_1)$ is a (μ, λ)-semilinear isomorphism.

Let

$$A_i = \begin{pmatrix} C & C_i \\ 0 & D \end{pmatrix} \ (i \text{ is odd}) \qquad \text{and} \qquad A_i = \begin{pmatrix} D & C_i \\ 0 & C \end{pmatrix} \ (i \text{ is even}),$$

the upper triangular matrix rings, and let

$$U_i = \begin{pmatrix} D & 0 \\ C_i & C \end{pmatrix} \ (i \text{ is odd}) \qquad \text{and} \qquad U_i = \begin{pmatrix} C & 0 \\ C_i & D \end{pmatrix} \ (i \text{ is even}).$$

Then by [9, Corollary 10.3], U_{i+1} becomes (A_{i+2}, A_i)-bimodules, which defines a Morita duality. By using λ, μ, ϕ, and ψ, we see that $A_6 \cong A_1$ and $A_7 \cong A_2$ as rings. Therefore we regard U_5 as an (A_1, A_4)-bimodule that defines a Morita duality. Similarly, we regard U_1 as an (A_2, A_5)-bimodule that defines a Morita duality. Now let

$$R = \begin{pmatrix} A_5 & U_4 & 0 & 0 & 0 \\ 0 & A_3 & U_2 & 0 & 0 \\ 0 & 0 & A_1 & U_5 & 0 \\ 0 & 0 & 0 & A_4 & U_3 \\ U_1 & 0 & 0 & 0 & A_2 \end{pmatrix}.$$

Using the dimensions a_i and b_i of C_i, we see that all A_i $(1 \le i \le 5)$ are pairwise non-isomorphic. Therefore by Proposition 2.2 R is a QF ring that does not have a Nakayama automorphism. R is the ring of [6, Remark 6.6], which was given as an example of a QF ring without weakly symmetric self-duality.

EXAMPLE 2.2. Let A be an artinian ring with self-duality but without weakly symmetric self-duality and let ${}_A U_A$ be a bimodule that defines a self-duality. (See Examples 2.1 and 2.3.) Then by Proposition 2.2 the ring

$$R = \begin{pmatrix} A & U & 0 & \cdots & 0 & 0 \\ 0 & A & U & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & U \\ U & 0 & 0 & \cdots & 0 & A \end{pmatrix}$$

is a QF ring without Nakayama automorphism.

We give another example of a QF ring without Nakayama automorphism.

EXAMPLE 2.3. With the same setting as in Example 2.1, let

$$A = \begin{pmatrix} C & C_1 & 0 & 0 & 0 \\ 0 & D & C_2 & 0 & 0 \\ 0 & 0 & C & C_3 & 0 \\ 0 & 0 & 0 & D & C_4 \\ C_5 & 0 & 0 & 0 & C \end{pmatrix}, \qquad B = \begin{pmatrix} C & C_3 & 0 & 0 & 0 \\ 0 & D & C_4 & 0 & 0 \\ 0 & 0 & C & C_5 & 0 \\ 0 & 0 & 0 & D & C_6 \\ C_7 & 0 & 0 & 0 & C \end{pmatrix},$$

$$U = \begin{pmatrix} C & 0 & 0 & 0 & C_6 \\ C_2 & D & 0 & 0 & 0 \\ 0 & C_3 & C & 0 & 0 \\ 0 & 0 & C_4 & D & 0 \\ 0 & 0 & 0 & C_5 & C \end{pmatrix},$$

where we regard C_5 in A, C_6 in U, and C_7 in B as (C, C)-bimodules by using the ring isomorphism $\lambda^{-1}\colon C \to D$. Then by [6, Theorem 6.4] U becomes a (B, A)-bimodule that defines a Morita duality and there exists a ring isomorphism $B \to A$ given by

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 \\ 0 & 0 & e & f & 0 \\ 0 & 0 & 0 & g & h \\ j & 0 & 0 & 0 & i \end{pmatrix} \mapsto \begin{pmatrix} \lambda(g) & \phi(h) & 0 & 0 & 0 \\ 0 & \mu(i) & \psi(j) & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & c & d \\ f & 0 & 0 & 0 & e \end{pmatrix}.$$

Therefore we can regard U as an (A, A)-bimodule, which defines a self-duality. Let R be the trivial extension of A by U. Then R is a QF ring by [9, Theorem 10.7]. Let e_i' be the (i, i)-matrix unit of A and let e_i be the idempotent of R corresponding to e_i'. We denote by i a composition factor that is isomorphic to $T(e_i R)$ or to $T(Re_i)$ $(i = 1, 2, \ldots, 5)$. By using the dimensions a_i and b_i of C_i, we see that the composition diagrams of the Loewy factors of the indecomposable projective modules of R_R and ${}_RR$ are the following ones.

R_R

1	2	3	4	5
2 2 2 3 3	3 4	4 4 5 5 5	5 5 1	1 2 2
4	5	1	2	3

${}_R R$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4\ 4\ 4\ 5\ 5 & 5\ 1 & 1\ 1\ 2\ 2\ 2 & 2\ 2\ 3 & 3\ 4\ 4 \\ 3 & 4 & 5 & 1 & 2 \end{array}$$

Thus the Nakayama permutation of R is $e_i \mapsto e_{[i+3]}$ and is cyclic. However, the lengths of $e_1 R$ and $e_4 R$ are different. Therefore there does not exist an automorphism τ of R such that $\tau(e_1) = e_4$ and hence R does not have a Nakayama automorphism.

3. H-RINGS WITHOUT SELF-DUALITY

In this section, by using the examples of QF rings without a Nakayama automorphism in the previous section and by giving a necessary and sufficient condition for a special type of H-rings to have a self-duality, we provide concrete examples of H-rings without self-duality.

A left artinian ring R is said to be a *left H-ring* (*Harada ring*) if there exists a basic set $\{e_{ij} | 1 \le i \le m,\ 1 \le j \le n(i)\}$ of orthogonal primitive idempotents for R satisfying the following conditions:

(1) $e_{i1} R_R$ is injective for each $i = 1, 2, \ldots, m$.

(2) $e_{ij} R \cong J(e_{i,j-1} R_R)$ for each $j = 2, 3, \ldots, n(i)$.

Right H-rings are defined similarly (cf. [5]). If R is a left H-ring, then by [5, Proposition 3.2] a minimal injective cogenerator $\bigoplus_{i,j} E(T(Re_{ij}))$ is finitely generated. Therefore we note that R is left Morita dual to $\operatorname{End}_R(\bigoplus_{i,j} E(T(Re_{ij})))$ by [1, Theorem 30.4].

As we mentioned in the Introduction, in [5] it is shown that (A), (B), and (C) are equivalent. Therefore, since (A) and (B) do not hold from the examples in Section 2, there must exist a left H-ring without self-duality.

The next theorem, which can be seen as a special case of [5, Proposition 3.3], gives rise to concrete examples of such rings. To state this, we use the following notation. Let R be a ring with a complete set I of orthogonal primitive idempotents. For a non-empty subset K of I, we put $e_K = \sum_{e \in K} e$. For a nonzero idempotent e of R, we define

$$R_e = \begin{pmatrix} eRe & eRe & eR(1-e) \\ J(eRe) & eRe & eR(1-e) \\ (1-e)Re & (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Then R_e is a ring by usual matrix operations. We note that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix} \cong R.$$

THEOREM 3.1. *Let R be a basic QF ring and let I be a complete set of orthogonal primitive idempotents for R with the Nakayama permutation σ on I. For a non-empty subset K of I, let $e = e_K$ and $e' = e_{\sigma(K)}$. Then*

(1) *R_e is a left and right H-ring.*

(2) *R_e is left Morita dual to $R_{e'}$.*

(3) *R_e has a self-duality if and only if there exists a ring automorphism τ of R such that $\tau(e) = e'$.*

Proof. (1) Let $A = eRe$, $B = (1-e)R(1-e)$, $U = eR(1-e)$, $V = (1-e)Re$, and $\Lambda = R_e$. Then

$$R = \begin{pmatrix} A & U \\ V & B \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} A & A & U \\ J(A) & A & U \\ V & V & B \end{pmatrix}.$$

Let I_A and I_B be complete sets of orthogonal primitive idempotents for A and B, respectively. We regard I as the disjoint union $I_A \dot\cup I_B$ and the Nakayama permutation σ as a permutation on $I_A \dot\cup I_B$. For $X, Y \in \{A, B\}$, let $I_{XY} = \{f \in I_X | \sigma(f) \in I_Y\}$. Then $I_A = I_{AA} \dot\cup I_{AB}$ and $I_B = I_{BA} \dot\cup I_{BB}$. For $f \in I_A$ and $g \in I_B$, let

$$f^{(1)} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad g^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g \end{pmatrix}.$$

Since R is a basic QF ring, we have

$$J(R) = \begin{pmatrix} J(A) & U \\ V & J(B) \end{pmatrix}, \qquad J(\Lambda) = \begin{pmatrix} J(A) & A & U \\ J(A) & J(A) & U \\ V & V & J(B) \end{pmatrix}.$$

Therefore, letting

$$S(R) = l_R(J(R)) = r_R(J(R)) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

since $S(\Lambda_\Lambda) = l_\Lambda(J(\Lambda))$ and $S({}_s\Lambda) = r_\Lambda(J(\Lambda))$, we have

$$S(\Lambda_\Lambda) = \begin{pmatrix} 0 & S_{11} & S_{12} \\ 0 & S_{11} & S_{12} \\ 0 & S_{21} & S_{22} \end{pmatrix}, \qquad S({}_\Lambda\Lambda) = \begin{pmatrix} S_{11} & S_{11} & S_{12} \\ 0 & 0 & 0 \\ S_{21} & S_{21} & S_{22} \end{pmatrix}.$$

By using these representations of socles, it follows that $(f^{(1)}\Lambda, \Lambda g^{(2)})$ $(f \in I_{AA})$, $(f^{(1)}\Lambda, \Lambda g^{(3)})$ $(f \in I_{AB})$, $(f^{(3)}\Lambda, \Lambda g^{(2)})$ $(f \in I_{BA})$, and $(f^{(3)}\Lambda, \Lambda g^{(3)})$ $(f \in I_{BB})$ are i-pairs, where $g = \sigma(f)$. Therefore, $f^{(1)}\Lambda_\Lambda$ $(f \in I_A)$, $f^{(3)}\Lambda_\Lambda$ $(f \in I_B)$, ${}_\Lambda\Lambda f^{(2)}$ $(f \in I_A)$, and ${}_\Lambda\Lambda f^{(3)}$ $(f \in I_B)$ are injective. Also it is clear that $f^{(2)}\Lambda_\Lambda \cong J(f^{(1)}\Lambda_\Lambda)$ and ${}_\Lambda\Lambda f^{(1)} \cong J({}_\Lambda\Lambda f^{(2)})$ for $f \in I_A$. Therefore Λ is a left and right H-ring.

(2) We first compute a minimal injective cogenerator for Λ-Mod. For $f \in I$, let $g = \sigma(f)$. It follows from the i-pairs of the proof of (1) above and [5, Proposition 3.2] that

$$E\big(T(\Lambda f^{(1)})\big) \cong \begin{cases} \Lambda g^{(2)} & (f \in I_{AA}), \\ \Lambda g^{(3)} & (f \in I_{AB}), \end{cases}$$

$$E\big(T(\Lambda f^{(2)})\big) \cong \begin{cases} \Lambda g^{(2)}/S(\Lambda g^{(2)}) & (f \in I_{AA}), \\ \Lambda g^{(3)}/S(\Lambda g^{(3)}) & (f \in I_{AB}), \end{cases}$$

$$E\big(T(\Lambda f^{(3)})\big) \cong \begin{cases} \Lambda g^{(2)} & (f \in I_{BA}), \\ \Lambda g^{(3)} & (f \in I_{BB}). \end{cases}$$

For $X, Y \in \{A, B\}$, let $f_{XY} = \sum_{f \in I_{XY}} f$ and $g_{XY} = \sum_{f \in I_{XY}} \sigma(f)$. Note that $f_{XY} \in X$ and $g_{XY} \in Y$. Furthermore, let $g_A = g_{AA}^{(2)} + g_{AB}^{(3)}$ and $g_B = g_{BA}^{(2)} + g_{BB}^{(3)}$, and let $E^{(1)} = E(T(\Lambda 1_A^{(1)}))$, $E^{(2)} = E(T(\Lambda 1_A^{(2)}))$, and $E^{(3)} = E(T(\Lambda 1_B^{(3)}))$. Then from the isomorphisms above we have $E^{(1)} \cong \Lambda g_A$, $E^{(2)} \cong \Lambda g_A/S(\Lambda g_A)$, and $E^{(3)} \cong \Lambda g_B$. Since $\{1_A^{(1)}, 1_A^{(2)}, 1_B^{(3)}\}$ is a complete set of orthogonal idempotents for Λ, $E^{(1)} \oplus E^{(2)} \oplus E^{(3)}$ is a minimal injective cogenerator for Λ-Mod.

From $1_A = g_{AA} + g_{BA}$, $1_B = g_{AB} + g_{BB}$, and the definition of e', we now note that

$$e' = \begin{pmatrix} g_{AA} & 0 \\ 0 & g_{AB} \end{pmatrix}, 1 - e' = \begin{pmatrix} g_{BA} & 0 \\ 0 & g_{BB} \end{pmatrix} \qquad \text{in } R = \begin{pmatrix} A & U \\ V & B \end{pmatrix}.$$

Then by using the isomorphisms of $E^{(i)}$ above and the fact that $r_R(S(R)) = J(R)$, it is routine to check that

$$\begin{aligned} &\mathrm{End}_\Lambda(E^{(1)} \oplus E^{(2)} \oplus E^{(3)}) \\ &\quad\cong \begin{pmatrix} \mathrm{Hom}_\Lambda(E^{(1)}, E^{(1)}) & \mathrm{Hom}_\Lambda(E^{(1)}, E^{(2)}) & \mathrm{Hom}_\Lambda(E^{(1)}, E^{(3)}) \\ \mathrm{Hom}_\Lambda(E^{(2)}, E^{(1)}) & \mathrm{Hom}_\Lambda(E^{(2)}, E^{(2)}) & \mathrm{Hom}_\Lambda(E^{(2)}, E^{(3)}) \\ \mathrm{Hom}_\Lambda(E^{(3)}, E^{(1)}) & \mathrm{Hom}_\Lambda(E^{(3)}, E^{(2)}) & \mathrm{Hom}_\Lambda(E^{(3)}, E^{(3)}) \end{pmatrix} \end{aligned}$$

$$\cong \begin{pmatrix} e'Re' & e'Re' & e'R(1-e') \\ J(e'Re') & e'Re' & e'R(1-e') \\ (1-e')Re' & (1-e')Re' & (1-e')R(1-e') \end{pmatrix} = R_{e'}.$$

(3) $(\Rightarrow)$ Assume that Λ has a self-duality. Let $A' = e'Re'$, $B = (1-e')R(1-e')$, $U' = e'R(1-e')$, $V' = (1-e')Re'$, and $\Lambda' = R_{e'}$. Then

$$\Lambda' = \begin{pmatrix} A' & A' & U' \\ J(A') & A' & U' \\ V' & V' & B' \end{pmatrix}.$$

By (2) there exists a ring isomorphism $\rho\colon \Lambda \to \Lambda'$. Let $I_{A'}$ and $I_{B'}$ be complete sets of orthogonal primitive idempotents for A' and B', respectively, and let

$$L = \{f^{(1)} | f \in I_A\} \mathbin{\dot\cup} \{f^{(2)} | f \in I_A\} \mathbin{\dot\cup} \{f^{(3)} | f \in I_B\},$$
$$L' = \{f'^{(1)} | f' \in I_{A'}\} \mathbin{\dot\cup} \{f'^{(2)} | f' \in I_{A'}\} \mathbin{\dot\cup} \{f'^{(3)} | f' \in I_{B'}\}.$$

Then L and L' are complete sets of orthogonal primitive idempotents for Λ and Λ', respectively. By [7, p. 42] we may assume that $\rho(L) = L'$. Since the injectivities of indecomposable projective Λ'-modules are similar to Λ in the proof of (1) and ρ preserves injective one-sided ideals, we must have $\rho(\{f^{(1)} | f \in I_A\}) = \{f'^{(1)} | f' \in I_{A'}\}$ and $\rho(\{f^{(2)} | f \in I_A\}) = \{f'^{(2)} | f' \in I_{A'}\}$. Thus, $\rho(1_A^{(1)}) = 1_{A'}^{(1)}$ and $\rho(1_A^{(2)}) = 1_{A'}^{(2)}$. Therefore ρ induces a ring isomorphism

$$\rho' : \begin{pmatrix} A & U \\ V & B \end{pmatrix} \to \begin{pmatrix} A' & U' \\ V' & B' \end{pmatrix} \qquad \text{such that } \rho' \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_{A'} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then it follows from the definitions of these two rings that ρ' induces a ring automorphism τ of R such that $\tau(e) = e'$.

$(\Leftarrow)$ Assume that there exists a ring automorphism τ of R such that $\tau(e) = e'$. Then we define $\rho\colon R_e \to R_{e'}$ via $\rho((r_{ij})_{i,j}) = (\tau(r_{ij}))_{i,j}$. Since $\tau(e) = e'$, ρ is well defined and is a ring isomorphism. Therefore by (2) R_e has a self-duality. ∎

Remark 3.1. In the theorem above, the ring R_e does not have a self-duality; that is, $R_e \not\cong R_{e'}$ in general. However, since the Nakayama permutation σ has a finite order, by iterating to take left Morita dual rings, we return to the first ring R_e.

The following corollary is a special case of Theorem 3.1.

COROLLARY 3.1. *Let R be a basic QF ring and let I be a basic set of* *orthogonal primitive idempotents for R with the Nakayama permutation σ on I. Assume that there exists a disjoint union $I = I_1 \,\dot\cup\, U_2 \,\dot\cup\, \cdots \,\dot\cup\, I_m$ such that $\sigma(I_i) = I_{[i+1]}$ for each i. Let $e_i = e_{I_i}$, $R_{ij} = e_i R e_j$, and*

$$\Lambda = \begin{pmatrix} R_{11} & R_{11} & R_{12} & \cdots & R_{1m} \\ J(R_{11}) & R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{m1} & R_{m1} & R_{m2} & \cdots & R_{mm} \end{pmatrix}.$$

Then

(1) *Λ is a left and right H-ring.*

(2) *Λ is left Morita dual to*

$$\Gamma = \begin{pmatrix} R_{22} & R_{22} & R_{23} & \cdots & R_{2m} & R_{21} \\ J(R_{22}) & R_{22} & R_{23} & \cdots & R_{2m} & R_{21} \\ R_{32} & R_{32} & R_{33} & \cdots & R_{3m} & R_{31} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{m2} & R_{m2} & R_{m3} & \cdots & R_{mm} & R_{m1} \\ R_{12} & R_{12} & R_{13} & \cdots & R_{1m} & R_{11} \end{pmatrix}.$$

(3) *Λ has a self-duality if and only if there exists a ring automorphism τ of R such that $\tau(e_1) = e_2$.*

Proof. (1) and (3). By noting that $\Lambda = R_{e_1}$ and $\Gamma = R_{e_2}$ and with the definition of e_i, these are special cases of Theorem 3.1 (1) and (3).

(2) Let f_0 be the idempotent of Λ such that its $(1,1)$-entry is 1_{A_1} and all other entries are zero, and for $i = 1, 2, \ldots, m$, let f_i be the idempotent of Λ such that its $(i+1, i+1)$-entry is 1_{A_i} and all other entries are zero. As well as the proof of Theorem 3.1, it follows that $E(T(\Lambda f_0)) \cong \Lambda f_2$, $E(T(\Lambda f_1)) \cong \Lambda f_2 / S(\Lambda f_2)$, $E(T(\Lambda f_2)) \cong \Lambda f_3, \ldots, E(T(\Lambda f_{m-1})) \cong \Lambda f_m$, and $E(T(\Lambda f_m)) \cong \Lambda f_1$. By using these isomorphisms, it is routine to check that the endomorphism ring Γ of a minimal injective cogenerator $\bigoplus_{i=0}^{m} E(T(\Lambda f_i)))$ has the desired form. ∎

Concluding this paper, we give two concrete examples of H-rings without self-duality.

EXAMPLE 3.1. Let A_i and U_i $(i = 1, 2, \ldots, 5)$ be the same as in Example 2.1. Therefore ${}_{A_{[i+2]}}U_{[i+1]A_i}$ defines a Morita duality and all

$A_1, A_2, \ldots, A_5$ are pairwise non-isomorphic. Let

$$\Lambda = \begin{pmatrix} A_5 & A_5 & U_4 & 0 & 0 & 0 \\ J(A_5) & A_5 & U_4 & 0 & 0 & 0 \\ 0 & 0 & A_3 & U_2 & 0 & 0 \\ 0 & 0 & 0 & A_1 & U_5 & 0 \\ 0 & 0 & 0 & 0 & A_4 & U_3 \\ U_1 & U_1 & 0 & 0 & 0 & A_2 \end{pmatrix}.$$

Since A_5 and A_3 are not isomorphic, Λ is a left and right H-ring without self-duality by Corollary 3.1.

Since each A_i is a 2×2 matrix ring, Λ is a 12×12 matrix ring. Let e_i be the (i, i)-matrix unit and we denote by i a composition factor that is isomorphic to $T(e_i\Lambda)$ or to $T(\Lambda e_i)$ $(i = 1, 2, \ldots, 12)$. Then the composition diagrams of the Loewy factors of the indecomposable projective modules of Λ_Λ and ${}_\Lambda\Lambda$ are the following.

Λ_Λ

1	2	3	4	5	6
3	4	2	5 5	6 6	7
2	5 5	4	6	7	8
4	6	5			
5					

7	8	9	10	11	12
8 8 8	9	10 10	11 11	12	1 1 1
9	10	11	12	1	3 3 3
				3	2
					4

${}_\Lambda\Lambda$

1	2	3	4	5	6
12	3 3	1	2	4 4	5
11	1 1	12	3 3	2 2	4
	12	11	1 1	3	2
			12	1	

7	8	9	10	11	12
6 6 6	7	8 8	9 9	10	11 11 11
5	6	7	8	9	10

EXAMPLE 3.2. Let R be the QF ring without a Nakayama automorphism in Example 2.3 and let $e_1, e_2, \dots, e_5$ be the primitive idempotents for R. The Nakayama permutation of R is $e_i \mapsto e_{[i+3]}$ and is cyclic. Let

$$\Lambda = R_{e_1} = \begin{pmatrix} e_1Re_1 & e_1Re_1 & e_1R(1-e_1) \\ J(e_1Re_1) & e_1Re_1 & e_1R(1-e_1) \\ (1-e_1)Re_1 & (1-e_1)Re_1 & (1-e_1)R(1-e_1) \end{pmatrix}.$$

Then Λ has six isomorphism classes of the simple modules. By example 2.3 the composition lengths of $e_1\Lambda$ and $e_4\Lambda$ are different. Therefore by Theorem 3.1 Λ is a left and right H-ring without self-duality.

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